On two-dimensional inertial flow in a rotating stratified fluid

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An inviscid fluid is bounded above by a horizontal plane and below by finite-amplitude ridges. The fluid is rotating and stratified. A uniform transport is forced across the ridges at small dynamical Rossby number, although the boundary conditions are such that motion cannot remain geostrophic. The most significant parameter is found to be a thermal Rossby number based upon the vertical density difference and both vertical and horizontal length scales, but independent of the transport. Conditions determining whether or not effects of the bottom topography will penetrate vertically throughout the fluid are discussed. Some numbers characteristic of flow in the deep ocean are presented.

1. Introduction

The constraint imposed upon the motion of a fluid relative to a system of rotating co-ordinates is such as to inhibit variations of velocity in the direction of the rotation vector. Let this be the vertical direction. Then the slow, steady motion of an inviscid fluid consists of a geostrophic balance between the horizontal pressure gradient and Coriolis accelerations, while the vertical pressure gradient remains in hydrostatic equilibrium. If the fluid is of homogeneous density, vertical variation of the velocity is completely prohibited; if inhomogeneous, the vector product of the rotation and the vertical gradient of the horizontal velocity must equal the horizontal gradient of the differential gravitational force. The field of motion thus restricted can satisfy only very special boundary conditions. In particular, it may not be consistent with the physical requirement of zero normal velocity upon an arbitrary rigid surface. In such a case, if the motion is to remain steady, the velocity or velocity gradient must become large, at least somewhere in the flow. Thus a boundary condition inconsistent with the strongly rotationally constrained motion is in the nature of a singular perturbation on the flow. As such, boundary effects may be confined, and thus only of local interest, or they may be of importance everywhere throughout the fluid.

The essential features of the mean flow in the deep sea have been discussed in terms of a theoretical model which is geostrophic and hydrostatic, and which indicates that the horizontal and vertical velocities are of the order of 10^{-2} and 10^{-5} cm sec^{-1}, respectively (Robinson & Stommel 1959). The vertical velocity, although extremely small, is considered to control the mean flow below the main thermocline (Stommel & Arons 1960). It is thus of importance to consider other mechanisms, not included in the simple theoretical model, capable of producing vertical velocity in the deep sea, and to inquire into their effect upon the mean flow. It is particularly relevant to do so now that direct velocity measurements at great depths have become possible by means of a neutrally buoyant float which may be tracked audibly for many months (Swallow 1957). The simple mechanism considered here is that of two-dimensional bottom topography.

We consider the motion of a fluid, which is bounded above by a horizontal plane and below by very long ridges, past which a uniform transport is forced at small Rossby number. Since the fluid is inviscid, the upper boundary could equally well be a free surface with very small gradients. First, we consider the topography to be neither steep nor high, resulting in flows of a type first discussed by Queney for application to the atmosphere (see, for example, Corby 1954), our results differing because of the boundary condition associated with finite rather than infinite vertical extent. The results of the linearized analysis are then used to infer an appropriate finite-amplitude model. The basic difference between the problem considered here and that of atmospheric lee waves is that rotational effects are here considered dominant rather than negligible. This is because the relatively weak oceanic flows correspond to a much smaller Rossby number for a given topographic scale. The inviscid results are of some interest for the general theory of rotating fluids, as departures from geostrophy dominated by frictional processes have been more intensively studied.

2. Formulation

Consider the steady motion of a fluid relative to a rotating co-ordinate frame in which the rotation vector $\mathbf{\Omega}$ is anti-parallel to gravity. By assumption, all processes which diffuse momentum and heat are to have negligible effect on the features of the flow to be considered, i.e. if the flow is turbulent, the mean fields are assumed to be essentially independent of turbulent transfer. The equation of state of the fluid is taken to be a linear dependence of density on temperature only; furthermore, the thermal expansion coefficient α is taken to be identically zero except when coupled with the gravitational acceleration g. Consistently, the ratio of centrifugal to gravitational accelerations is assumed vanishingly small. In summary, we consider, under the Boussinesq approximation, an ideal fluid which is subject to Coriolis accelerations.

The equations of conservation of momentum, mass and heat take the form

$$\mathbf{v}'.\nabla'\mathbf{v}' + 2\Omega\mathbf{k}\times\mathbf{v}' - \alpha g T'\mathbf{k} + \rho_0^{-1}\nabla p' = 0, \tag{2.1}$$

$$\nabla'.\mathbf{v}' = 0, \tag{2.2}$$

$$\mathbf{v}'.\nabla' T' = 0, \tag{2.3}$$

where $(u', v', w') = \mathbf{v}'$ are the velocity components in the (x', y', z') directions, T' is the temperature, p' is the pressure minus the hydrostatic pressure due to the mean density ρ_0, and $\mathbf{k}$ is a unit vector in the vertical z'-direction.

The fluid is taken to be of finite vertical extent, but unbounded horizontally. The upper rigid surface is taken to be the horizontal plane $z_0' = h$, a constant; the lower rigid surface is taken to be a function of one horizontal co-ordinate,

$z_1' = B(y')$. For convenience, this surface is placed so that the horizontally averaged value of B is zero. The corresponding kinematic boundary conditions are

$$w'(x', y', h) = 0, \tag{2.4}$$

$$w'(x', y', B(y')) = \frac{dB}{dy'} v'(x', y', B(y')). \tag{2.5}$$

The influence of these conditions on the flow is to be the primary object of study.

Fluid motion is to be maintained by an external pressure gradient which forces a net transport in the y'-direction which is everywhere uniform and equal to $V_0 h$. Because of the rotation, the horizontal pressure gradient which drives the motion must be in the x'-direction, i.e. perpendicular to the direction of the transport. We shall consider, however, only two-dimensional responses to the driving force, fields of velocity and temperature which are uniform along the ridges. Formally, let

$$p' = \rho_0 2\Omega V_0[x' + k^{-1}p(y, z)], \tag{2.6}$$

$$\mathbf{v}' = V_0 \mathbf{v}(y', z'), \tag{2.7}$$

$$T' = T_0 . T(y', z'), \tag{2.8}$$

where k is the wave-number, or inverse of the characteristic scale, of the bottom surface, and T_0 is the total temperature difference between the upper and lower surfaces. The fluid is taken to be stably stratified. Correspondingly we require the non-dimensional temperature T to be $\pm\frac{1}{2}$ on the top and bottom surfaces, respectively. If $dB/dy' = 0$, there would be a geostrophic flow, $v = 1$, $u = w = 0$, independent of the thermal structure. We are, therefore, considering the two-dimensional modification due to bottom topography of an otherwise barotropic flow (for p, T). However, since $dB/dt' \neq 0$, the actual flow will be baroclinic.

Introducing the non-dimensional independent variables $y = ky'$, $z = h^{-1}z'$, and isolating the functional form of the lower boundary from its amplitude by setting $B(y') = b_0 b(y)$, where b has unit amplitude, we substitute (2.6–8) into the equations and boundary condition (2.5). The equations, which now expose the relevant non-dimensional parameters, become

$$\epsilon[vu_y + \lambda w u_z] - v + 1 = 0, \tag{2.9}$$

$$\epsilon[vv_y + \lambda w v_z] + u + p_y = 0, \tag{2.10}$$

$$\epsilon^2[\lambda^{-1} v w_y + w w_z] - \tau T + \epsilon p_z = 0, \tag{2.11}$$

$$v_y + \lambda w_z = 0, \tag{2.12}$$

$$vT_y + \lambda w T_z = 0; \tag{2.13}$$

and the boundary condition becomes

$$w(y, \beta b(y)) = \delta \frac{db}{dy} v(y, \beta b(y)). \tag{2.14}$$

Here, we have defined

$\epsilon \equiv V_0 k(2\Omega)^{-1}$ an inertial parameter (dynamical Rossby number),

$\tau \equiv \alpha g T_0 h k^2 (2\Omega)^{-2}$ a rotational-stratification parameter (thermal Rossby number), and the geometric parameters

$\lambda \equiv (kh)^{-1}$ the horizontal-to-vertical scale ratio,

$\beta \equiv b_0 h^{-1}$ the topographic height,

$\delta \equiv b_0 k$ the topographic steepness.

Of the last two, only one is independent, but it is useful to define them both.

The apparent direct measure of all non-linear effects is the dynamical Rossby number ϵ, as $b(y)$ appearing in (2.14) is a given function. However, even if ϵ is small, an ordinary perturbation expansion cannot be made about the non-linear terms, for, as can be seen from (2.9) and (2.12), all vertical variation in v and w is prohibited in the geostropic first approximation. Thus a singular perturbation expansion is necessary; to overcome this constraint, large velocities or velocity gradients must occur somewhere or everywhere in the flow.

The problem can, however, be consistently linearized with respect to the topographic steepness parameter δ, for as the gradient of the bounding surface vanishes no vertical variation in the flow is necessary. In order to gain insight into the role of the various parameters, we shall first treat this linearized problem. This will be of particular value because the parameters are numerous, and the uniform validity of an approximation in one must be strictly limited in terms of the others. The information obtained from the linearized problem will then be used to develop a useful finite-amplitude approximation.

3. The linearized problem

With the assumption that the topographic gradients are not steep, $\delta < 1$, an ordinary perturbation expansion of the form

$$\left.\begin{aligned} v = 1 + \sum_{n=1}^{\infty} \delta^n v_n, \quad T = (z - \tfrac{1}{2}) + \sum_{n=1}^{\infty} \delta^n T_n, \\ u = \sum_{n=1}^{\infty} \delta^n u_n, \quad w = \sum_{n=1}^{\infty} \delta^n w_n, \quad p = \sum_{n=1}^{\infty} \delta^n p_n, \end{aligned}\right\} \tag{3.1}$$

can be made. Note that the basic temperature field is taken to have a constant vertical gradient, although any function of z alone is consistent with the expansions. The contributions of order δ to equations (2.9–13), the equations for the first-order fields, are

$$\epsilon u_{1y} - v_1 = 0, \tag{3.2}$$

$$\epsilon v_{1y} + u_1 + p_{1y} = 0, \tag{3.3}$$

$$\epsilon^2 \lambda^{-1} w_{1y} - \tau T_1 + \epsilon p_{1z} = 0, \tag{3.4}$$

$$v_{1y} + \lambda w_{1z} = 0, \tag{3.5}$$

$$T_{1y} + \lambda w_1 = 0. \tag{3.6}$$

Cross-differentiation yields the equation for a single function, for example, w_1, which is

$$\left\{\left(\epsilon^2 \frac{\partial^2}{\partial y^2} + 1\right) \frac{\partial^2}{\partial z^2} + \left(\tau + \epsilon^2 \lambda^{-2} \frac{\partial^2}{\partial y^2}\right) \frac{\partial^2}{\partial y^2}\right\} w_1 = 0. \tag{3.7}$$

The first-order statement of the kinematic boundary condition (2.14) reduces to a condition on the vertical velocity alone

$$w_1(y, \beta b(y)) = \frac{db}{dy}. \tag{3.8}$$

The problem becomes particularly simple if the further and independent assumption is made that the topographic height is small, $\beta \ll 1$. This is equivalent to assuming that $\lambda \ll \delta^{-1}$. Now a power series expansion of (3.8) may be made about $z = 0$, and, if only the first term is retained, the condition becomes $w_1(y, 0) = db/dy$. Simple separated solutions of (3.8) are now possible. Furthermore, we shall be concerned only with the particular solution of (3.9), and not with the freedom associated with the high y-order. Since the problem has been linearized, the Fourier modes of the bottom surface may be considered separately. The equations separate with the horizontal component of the flow across the ridges, the pressure, and the temperature in phase, and the horizontal component along the ridges and the vertical component of the flow out of phase, with the bottom. As an example, consider $b = \sin y$, the wavelength being already contained in λ. The corresponding vertical solutions of (3.7) are

$$\exp\left\{\pm\sqrt{\left(\frac{\tau - \epsilon^2\lambda^{-2}}{1-\epsilon^2}\right)}z\right\}.$$

Note that the qualitative behaviour of the solutions depends strongly on the relative sizes of τ, ϵ, λ. To demonstrate this, two limiting cases are discussed below, both for weak basic flow, $\epsilon \ll 1$. Thus ϵ^2 will be neglected in the denominator of the radical in the argument of the exponential. This corresponds to neglecting the inertial term in (3.3). In the first case the fluid is to be homogeneous, in the second, to be effectively strongly stratified.

Case I. $\epsilon \ll 1$, $\tau = 0$

The solutions satisfying the prescribed conditions on the vertical velocity are

$$\left.\begin{aligned} u_1 &= \frac{\cos y \cos \epsilon\lambda^{-1}(1-z)}{\sin \epsilon\lambda^{-1}}, \quad v_1 = -\frac{\epsilon \sin y \cos \epsilon\lambda^{-1}(1-z)}{\sin \epsilon\lambda^{-1}}, \\ w_1 &= \frac{\cos y \sin \epsilon\lambda^{-1}(1-z)}{\sin \epsilon\lambda^{-1}}, \quad T_1 = -\frac{\lambda \sin y \sin \epsilon\lambda^{-1}(1-z)}{\sin \epsilon\lambda^{-1}}, \\ p_1 &= \frac{\sin y \cos \epsilon\lambda^{-1}(1-z)}{\sin \epsilon\lambda^{-1}}. \end{aligned}\right\} \tag{3.9}$$

In this case the solutions are in the form of inertial waves of natural vertical scale $\lambda\epsilon^{-1} = (2\Omega)(V_0 k^2 h)^{-1}$. Therefore the influence of the bottom topography penetrates throughout the homogeneous fluid, even though the topography is not steep or high. If the natural scale is much smaller than the geometrical scale, $\lambda \ll \epsilon$, many oscillations occur between the top and bottom surfaces. The appearance of $\sin \epsilon\lambda^{-1}$ in all denominators indicates possible resonance with arbitrarily large amplitudes as $\epsilon\lambda^{-1} \to n\pi$. Large gradients and amplitudes strongly restrict the validity of the initial expansion. Furthermore, eigenfunctions of the homogeneous problem, $w(0) = w(1) = 0$, exist and may be added to the solution with arbitrary amplitude.

If the natural scale is much larger than the geometrical scale, $\lambda \gg \epsilon$, less than one oscillation occurs; in the limit as $\epsilon\lambda^{-1}$ vanishes, u_1, v_1, p_1 are independent of z, while w_1, T_1 vary linearly. The only appearance of ϵ which remains is in the amplitudes of u_1 and p_1, which are $O(\epsilon^{-1}\lambda)$; a large flow along the ridges overcomes the rotational constraint. Explicitly, retaining the first two terms in the expansion of the cosine and the sine, we find

$$u_1 \sim \cos y\{\epsilon^{-1}\lambda + \epsilon\lambda^{-1}[\tfrac{1}{6} - \tfrac{1}{2}(1-z)^2]\}.$$

It should be remarked that the actual large vertically-constant velocity given by the first term is independent of the basic velocity, i.e.

$$u_1' \sim V_0\epsilon^{-1}\lambda\cos y = 2\Omega h^{-1}k^{-2}\cos y.$$

The fact that the flow along the ridges becomes very large suggests that major modification of the flow might occur if the fluid were finite instead of infinite in this direction; boundary conditions upon u might significantly alter the motion. To investigate this effect we have considered the flow in an infinite channel with a wavy bottom, i.e. the same problem as above but containing the fluid in $-\tfrac{1}{2} \leqslant x \equiv x'/d \leqslant \tfrac{1}{2}$. The three-dimensional operator which now replaces (3.7) with $\tau = 0$ is

$$\left\{\epsilon^2\left[(kd)^{-2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda^2\frac{\partial^2}{\partial z^2}\right]\frac{\partial^2}{\partial y^2} + \lambda^2\frac{\partial^2}{\partial z^2}\right\}w_1 = 0, \tag{3.10}$$

and the solution for u_1 after the satisfaction of all boundary conditions is, for $\epsilon \ll 1$,

$$u_1 = \cos y\left\{\frac{\cos\epsilon\lambda^{-1}(z-1)}{\sin\epsilon\lambda^{-1}} + \epsilon^{-1}\lambda\left[-1 + 2\sum_{m=1}^{\infty}\frac{\cos\mu(d/h)\,x\cos m\pi z}{\mu^2\cos\mu(d/2h)}\right]\right\}, \tag{3.11}$$

where

$$\mu = \sqrt{\left\{\left(\frac{m\pi}{\epsilon}\right)^2 - \frac{1}{\lambda^2}\right\}}.$$

The solution is seen to be that of an inertial wave in the x-direction also, the effect of the side walls penetrating throughout the fluid, and additional resonances being indicated. Taking the limit of $\epsilon\lambda^{-1} \ll 1$, as above, we find that

$$u_1 \sim \cos y\left\{\epsilon\lambda^{-1}\left[\tfrac{1}{6} - \tfrac{1}{2}(1-z)^2 + 2\sum_{m=1}^{\infty}\frac{\cos(m\pi/\epsilon)(d/h)\,x\cos m\pi z}{(m\pi)^2\cos(m\pi/\epsilon)(d/2h)}\right]\right\};$$

the large, $O(\epsilon^{-1}\lambda)$, contribution has entirely disappeared. What has happened is that, in the presence of p_{1x}, the fluid tends to remain geostrophic, but not hydrostatic; large pressure gradients overcome the rotational constraint now that the velocity component across the direction of the transport has been restricted.

Case II. $\epsilon \ll 1$, $\tau \gg \sqrt{(\epsilon\lambda^{-1})}$

The appropriate approximate form of the solutions to (3.2–6) are now

$$\left.\begin{aligned}
u_1 &= \frac{\lambda\tau^{\frac{1}{2}}}{\epsilon}\cos y\,\frac{\cosh\tau^{\frac{1}{2}}(1-z)}{\sinh\tau^{\frac{1}{2}}}, & v_1 &= -\frac{\lambda\tau^{\frac{1}{2}}\sin y\cosh\tau^{\frac{1}{2}}(1-z)}{\sinh\tau^{\frac{1}{2}}},\\
w_1 &= \frac{\cos y\sinh\tau^{\frac{1}{2}}(1-z)}{\sinh\tau^{\frac{1}{2}}}, & T_1 &= -\frac{\lambda\sin y\sinh\tau^{\frac{1}{2}}(1-z)}{\sinh\tau^{\frac{1}{2}}},\\
p_1 &= -\frac{\lambda\tau^{\frac{1}{2}}}{\epsilon}\frac{\sin y\cosh\tau^{\frac{1}{2}}(1-z)}{\sinh\tau^{\frac{1}{2}}}.
\end{aligned}\right\} \tag{3.12}$$

The solutions are no longer inertial waves, but contain a natural length scale based upon the thermal Rossby number τ only. The remaining role of the dynamic Rossby number ϵ is to set the amplitude of u_1 and p_1. The flow remains hydrostatic and departs from geostrophic only in the first equation of motion. The non-dimensional velocity components across the ridges and in the vertical are similar in τ alone, i.e. independent of V_0. If the natural length scale is larger than the geometrical scale, $\tau^{\frac{1}{2}} < 1$, the first-order fields fill smoothly the entire region between the upper and lower surfaces. If, however, the natural length scale is smaller than the geometrical, $\tau^{\frac{1}{2}} > 1$, the first-order fields become essentially confined within a region above the bottom surface, the width of which decreases with increasing τ. Thus when $\tau^{\frac{1}{2}} \gg 1$, the response to the bottom topography is an entirely local effect, the flow field and temperature modification occurring in a narrow boundary layer. In this interesting limit, the functional forms of the solutions (3.12) may be approximated by

$$\frac{\sinh \tau^{\frac{1}{2}}(1-z)}{\sinh \tau^{\frac{1}{2}}} \sim \frac{\cosh \tau^{\frac{1}{2}}(1-z)}{\sinh \tau^{\frac{1}{2}}} \sim e^{-\tau^{\frac{1}{2}}z}. \tag{3.13}$$

It should be noted, however, that in this boundary layer the horizontal velocities become very large, increasing as $\tau^{\frac{1}{2}}$; u_1 is also large as ϵ^{-1}, and the vertical, gradients are large as well. Thus severe restrictions again occur on the range of validity of the ordinary perturbation expansions in the topographic steepness and height, δ and β.

4. The finite amplitude problem

The mathematical model and analysis

Of the results of the linearized analysis, those of Case II hold the greatest interest for possible oceanographic application, where the dynamical Rossby numbers and the vertical-to-horizontal scale ratios are usually small, and the pressure tends to remain essentially hydrostatic. Furthermore, even if the vertical temperature gradient is only one degree in a thousand meters, the thermal Rossby number that we have defined will be large for topographic length scales up to about a 100 km. Thus the final result of an inertial-thermal boundary layer is most pertinent, but, as mentioned above, the large velocities and velocity gradients that occur limit the validity of the expansions to extremely small and smooth topographic elements.

To examine the limitations imposed, we consider the order of magnitude of terms neglected in equations (2.9–14) in arriving at the approximate solutions obtained by inserting (3.13) into (3.12). Note that u_1, p_1, are $O(\lambda\tau^{\frac{1}{2}}\epsilon^{-1})$, v_1 is $O(\lambda\tau^{\frac{1}{2}})$, w_1 is $O(1)$, T_1 is $O(\lambda)$, and $\partial/\partial z$ is $O(\tau^{\frac{1}{2}})$. Two types of terms have been neglected, second (and higher) order terms of the initial δ-expansion, and terms omitted because of subsequent assumptions made concerning ϵ, τ and λ (or β). Terms of the latter type will be designated type A, and those of the former, type B. The order of magnitude of the first neglected terms of these two types, relative to the magnitude of the retained terms, are tabulated as follows for the first, second, and third momentum equations, the heat equation, and the kinematic boundary condition, in that order.

equation no.	(2.9)	(2.10)	(2.11)	(2.13)	(2.14)
type A	none	ϵ^2	$\lambda^{-2}\tau^{-1}\epsilon^2$	none	$\delta\lambda\tau^{\frac{1}{2}}$
type B	$\delta\lambda\tau^{\frac{1}{2}}$	$\delta\lambda\tau^{\frac{1}{2}}\epsilon^2$	$\delta\lambda^{-1}\tau^{-\frac{1}{2}}\epsilon^2$	$\delta\lambda\tau^{\frac{1}{2}}$	$\delta\lambda\tau^{\frac{1}{2}}$

Recall that the conditions for validity of the solutions are roughly $\delta < 1$, $\epsilon < 1$, $\tau > 1$ and $\lambda < \delta^{-1}$ but $\lambda > \epsilon\tau^{-2}$. It is seen, therefore, that, except in the kinematic boundary condition (2.14), the approximations leading to the omission of terms of type A impose no new restrictions; they are good approximations for this range of parameters, even in the boundary layer. However, terms of type B in equations (2.9), (2.13), and both terms of (2.14), require that $\delta\lambda\tau^{\frac{1}{2}} < 1$, a strong restriction on the height or steepness of the bottom topography. The terms in (2.10) and (2.11) require nothing new. These results may be summarized by noting that more terms of the same type as those that have already contributed to the boundary-layer solutions become more and more important as the topography becomes larger or steeper, but terms which have not contributed (i.e. the inertial terms in the second and third momentum equations) continue to be negligible.

The preceding discussion suggests the following approximate non-linear model as appropriate for the range of parameters under consideration. The fluid motion is assumed to remain geostrophic in the horizontal direction normal to the gradient of the bottom surface, and the pressure is assumed to remain hydrostatic, but the full non-linear terms in the first momentum equation and the heat equation are to be retained, and the kinematic boundary condition is to be satisfied exactly. Returning to equations (2.9–13), these assumptions are formalized by assuming $\epsilon \ll 1$, and expanding in this parameter after recognizing that the leading contributions to u and p are $O(\epsilon^{-1})$, while the leading terms for the other fields are independent of ϵ. Defining

$$\mu \equiv \epsilon u, \quad \pi \equiv \epsilon p, \tag{4.1}$$

and introducing a stream function for v and w because of the two-dimensional form of (2.12) as

$$v = \psi_z, \quad w = -\lambda^{-1}\psi_y, \tag{4.2}$$

substituting these expressions into (2.9, 10, 11, 13), and neglecting contributions $O(\epsilon^2)$, we find that the approximate equations are

$$\psi_z \mu_y - \psi_y \mu_z - \psi_z + 1 = 0, \tag{4.3}$$

$$\mu + \pi_y = 0, \tag{4.4}$$

$$-\tau T + \pi_z = 0, \tag{4.5}$$

$$\psi_z T_y - \psi_y T_z = 0. \tag{4.6}$$

Furthermore, the pressure can be easily eliminated between (4.4, 5), and these equations replaced by the simple (thermal wind) relationship

$$\mu_z + \tau T_y = 0. \tag{4.7}$$

The boundary condition (2.14) is now satisfied exactly by requiring the bottom surface to be a streamline; unit transport in the positive y-direction is taken as

$$\psi(y, 1) = 0, \quad \psi(y, \beta b(y)) = -1. \tag{4.8}$$

The problem that has been posed may be solved most simply by a transformation of independent and dependent variables, which is suggested both by the form of the equations and by the nature of the boundary conditions. The variables z, μ, T are now considered as dependent, and ψ, y as independent. Under the condition $(\partial\psi/\partial z)_y \neq 0$, with subscripts hereafter referring to partial differentiation with respect to the new set of independent variables, (4.3, 7, 6) transform to

$$\mu_y + z_\psi - 1 = 0, \tag{4.9}$$

$$\mu_\psi - \tau T_\psi z_y = 0, \tag{4.10}$$

$$T_y = 0. \tag{4.11}$$

An additional term containing a factor T_y appears in the direct transformation of (4.7) into (4.10), which is zero in virtue of (4.11), i.e. since the temperature is dependent upon the stream function only.

Integrating (4.11) in the form $T = t(\psi)$, extracting a particular solution by defining $\zeta = z - \psi - 1$, substituting into (4.9, 10), and eliminating μ by cross-differentiation, yields the basic equation

$$\zeta_{\psi\psi} + \tau t'(\psi)\,\zeta_{yy} = 0, \tag{4.12}$$

which must be solved together with the boundary conditions

$$\zeta(0, y) = 0, \quad \zeta(-1, y) = \beta b(y). \tag{4.13}$$

Cast in this form, the non-linear problem is seen to have a remarkable property, viz. both the basic equation (4.12) and the boundary conditions (4.13) allow simple separation and solution by Fourier superposition. Thus if,

$$b(y) = \int_{-\infty}^{\infty} b_n\, e^{-iny}\, dn, \tag{4.14}$$

then

$$\zeta(\psi, y) = \int_{-\infty}^{\infty} f_n(\psi)\, e^{-iny}\, dn, \tag{4.15}$$

where

$$f''_n - \tau n^2 t'(\psi) f_n = 0 \tag{4.16}$$

and

$$f_n(0) = 0, \quad f_n(-1) = \beta b_n. \tag{4.17}$$

Arbitrarily shaped, finite-amplitude bottom topography may be analysed in terms of its component Fourier modes, and the flow (i.e. the position of a streamline) obtained by solving a linear ordinary differential equation, which will, in general, have a non-constant coefficient.

Discussion

To illustrate the method, and to obtain qualitative features of the results, the single mode $b = \sin y$ will again be considered. To proceed we must, of course, first determine the functional relationship between the temperature and the stream function. In this case we do so by requiring the temperature to reduce to a linear function of z alone when the topography becomes vanishingly small,

$\beta \to 0$. In other words, we pose the same problem as was posed in perturbation form in § 3 above, and the results must agree in the range of overlap of validity. This is seen to require $t(\psi) = \psi + \frac{1}{2}$. Thus $t'(\psi) = 1$, and the equation corresponding to (4.16) has constant coefficients and simple exponential solutions. This is, of course, analytically the simplest problem of its type. In general, the functional relation between T and ψ must be obtained by specifying both quantities as functions of z at some value of y. The solutions for the case considered are

$$\left.\begin{aligned} z &= 1+\psi-\beta\frac{\sin y \sinh \tau^{\frac{1}{2}}\psi}{\sinh \tau^{\frac{1}{2}}}, \\ \mu &= \beta\tau^{\frac{1}{2}}\frac{\cos y \cosh \tau^{\frac{1}{2}}\psi}{\sinh \tau^{\frac{1}{2}}}, \\ T &= \psi+\tfrac{1}{2}. \end{aligned}\right\} \tag{4.18}$$

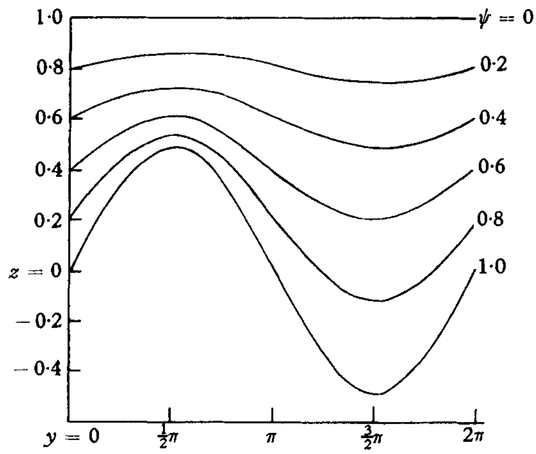

FIGURE 1. Streamlines for $\beta = \frac{1}{2}$, $\tau = 4$, $\epsilon \ll 1$.

The character of the solution may be seen in figure 1, where the streamlines are plotted for $\beta = \frac{1}{2}$ and $\tau = 4$. There is a symmetrical (about the crest and trough) distortion of the streamlines by the wavy bottom, which decreases towards the straight upper surface streamline.

For a fixed β, as τ increases, essentially straight streamlines obtain in a shorter distance above the bottom surface, i.e. the boundary layer over the crest, which may already be seen forming in figure 1, becomes narrower, and the flow over the crest becomes more rapid; correspondingly there is less distortion, and therefore less transport, in the trough. In this range, therefore, topographic effects influence the flow only locally. However, the behaviour for large τ ultimately limits the applicability of the theory, i.e. a relatively high crest blocks the flow when the fluid is effectively strongly stratified. The mathematical expression of this limitation is that z ceases to be a monotonic function of ψ, and thus the solution ceases to have physical meaning. From (4.18)

$$z_\psi = 1-\beta\tau^{\frac{1}{2}}\sin y\frac{\cosh \tau^{\frac{1}{2}}\psi}{\sinh \tau^{\frac{1}{2}}};$$

the second term is largest at $y = \frac{1}{2}\pi$ (the crest), and the condition for validity is seen to be

$$\beta\tau^{\frac{1}{2}}\operatorname{ctnh}\tau^{\frac{1}{2}} < 1; \quad \text{or, approximately} \quad \beta < \tau^{-\frac{1}{2}} \quad \text{for} \quad \tau \gg 1. \tag{4.19}$$

To overcome this constraint, additional inertial accelerations, from the second or third momentum equations, must become important, at least in a limited region near the top of the crest. Solutions (4.18) may possibly remain valid away from $y = \frac{1}{2}\pi$, and an inertial boundary layer, dependent upon both ϵ and τ, govern the flow in a limited region near the top of the crest. Further consideration suggests, however, that solutions (4.18) may not be physically realized as condition (4.19) is approached, because the flow may become thermally unstable as the isotherms become steeply tilted. The preferred flow would conceivably consist of all the transport occurring in a warmer upper layer. Perhaps the greatest usefulness of condition (4.19) is as a criterion for the design of a critical experiment, increasing τ towards β^{-2} for various ϵ. Since τ depends upon Ω, and ϵ upon Ω and V_0, the variations could both be made mechanically.

Returning to a consideration of (4.18) and figure 1, for fixed β and decreasing τ, the topographic effects are seen to influence the flow everywhere. The solutions take particularly simple form when τ is much less than one, but still large enough for the fluid to remain hydrostatic. Expanding the functional form of the solution in this range and dropping terms $O(\tau)$, we find

$$z \sim 1 + (1 - \beta\sin y)\psi, \quad \mu = \epsilon u \sim \beta\cos y. \tag{4.20}$$

Thus the remaining role of τ is merely to determine the vertical distribution of pressure; buoyancy effects no longer influence the fluid motion. A large, $O(\epsilon^{-1})$, vertically constant cross-flow again overcomes the rotational constraint. If the basic vertical thermal gradient, $T_0 h^{-1}$, and the rotation, Ω, are fixed, the limit of small τ is associated with small wavelengths. To retain geophysical pertinence in the results as the horizontal topographic scale becomes large, account must be made for the variation of the effective rotation with latitude if y is a north–south co-ordinate. A variable Coriolis parameter may be introduced by replacing the constant Ω in (2.1) by $\Omega(1+\omega y)$, where ω measures the variation of the Coriolis effects with respect to the horizontal scale of the topographic elements. The solutions replacing (4.20), in terms of the general lower surface, are

$$z = 1 + (1 - \beta b(y))\psi, \quad \mu = \int [\beta b(y) - \omega y + \omega\beta y b(y)]\, dy. \tag{4.21}$$

Note that there is no change in the stream function for v, w, and that the variation of the Coriolis parameter and general bottom shape are symmetrical effects in their influence upon u. This is understandable, in that the variation of the Coriolis parameter is an expression of the curvature of the earth relative to the effective rotation vector (the component parallel to gravity).

If τ is not small, but ω effects are still important, the problem is not intrinsically more difficult, although it becomes more complicated analytically. ζ must now be written as a sum of terms of the form $\zeta_n = g_n(y) f_n(\psi)$, where f_n still satisfies (4.16), and g_n satisfies a Bessel equation, $g_n'' + n^2(1+\omega y)g_n = 0$, the solutions of which must be used to expand $b(y)$. The geophysical interest in this range is slight, as the conditions imply an excessively strong stratification.

To summarize the applicability of the qualitative results to the flow in the deep sea, numerical values of the parameters that are characteristic of oceanic conditions below the main thermocline are presented. The numerical values, computed for $T_0 = 5\,°C$, $\alpha = 2\times10^{-4}\,(°C)^{-1}$, $2\Omega = 10^{-4}\,(\text{sec})^{-1}$, $h = 5\times10^{5}\,\text{cm}$, are given in the following table.

V_0 (cm sec^{-1})	k (cm^{-1})	ϵ	τ	$\tau^{-\frac{1}{2}}$
1	2×10^{-7}	2×10^{-3}	2	7×10^{-1}
1	2×10^{-8}	2×10^{-4}	2×10^{-2}	7
10^{-2}	2×10^{-7}	2×10^{-5}	2	7×10^{-1}
10^{-2}	2×10^{-8}	2×10^{-6}	2×10^{-2}	7

The theory becomes useful when the horizontal scale is greater than 30 km, allowing a topographic height of 0·2. Features up to a scale of 100 km will not influence the basic flow significantly, but the effect of (sufficiently high) larger-scale topography will penetrate throughout the flow. It must be borne in mind that the stringent conditions of two-dimensionality and negligible influence of turbulent viscosity and conductivity must be met to make these results at all relevant.

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